Stochastic process of equilibrium fluctuations of a system with long-range interactions

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The relaxation towards equilibrium of systems with long-range interactions is not yet understood. As a step towards such a comprehension, we propose the study of dynamical equilibrium fluctuations in a model system with long-range interaction. We compute analytically, from the microscopic dynamics, the autocorrelation function of the order parameter. From this result, we derive analytically a Fokker-Planck equation which describes the stochastic process of the impulsion of a single particle in an equilibrium bath. The diffusion coefficient is explicitly computed.

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A number of physical systems are governed by long-range interactions. Some examples are given by self-gravitating systems, two-dimensional incompressible, or geophysical flows, some models in plasma physics. For such Hamiltonian systems, the nonadditivity of the interactions makes the usual thermodynamic limit $N\to\infty, V\to\infty$ irrelevant. Microcanonical average is, however, still relevant, and generically leads to a mean-field description of the equilibrium, exact in the limit $N\to\infty$ [1]. The relaxation toward equilibrium of these systems still has to be completely understood. The phenomenology of the dynamics shows that a rapid relaxation leads to the formation of quasistationary structures, which may be out of equilibrium states (see Ref. [2] for astrophysical and geophysical examples and Ref. [3] for spin system ones). In most cases, this is explained by the existence of stable stationary states of the associated Vlasov equation, which describes the dynamics by approximating the potential by a mean-field one. In such stable situations, the Vlasov dynamics is a good approximation of the particle dynamics, on typical time scales diverging with the number of particles [4]. The relaxation towards equilibrium of these structures is then associated to the fluctuations of the potential around its mean-field approximation, and is thus very slow. One of our goals is to understand such a relaxation, which is of particular interest, for instance in the study of astrophysical structures, turbulence parametrization in geophysical flows, etc. Some works towards a kinetic description of this relaxation have been proposed, for instance, by Chandrasekar in the context of self-gravitating systems [5], Chavanis for the point vortex model [6], or for the two-dimensional Euler equation [6], or in plasma physics [7]. In each of these cases, the relaxation is then described by a Fokker-Planck equation or some generalizations. The diffusion coefficient has been computed, in some limits, for the point vortex model [6] and for self-gravitating systems.

In the kinetic theory of dilute gases, the Boltzmann equation has led to the computation of transport coefficients [8], providing an example of explicit computation of a diffusion coefficient for a system, with a large number of particles. A complete mathematical proof of this result directly from the Hamiltonian dynamics is, however, still to be achieved. The computation of the diffusion coefficient for the standard map [9] is a classical example for a system with a small number of degrees of freedom. In the past decades, the issue of the link between chaotic Hamiltonian dynamics and diffusive properties has been addressed on a general framework [10]. We also note works on the relaxation to equilibrium of a massive piston in interaction with two out-of-equilibrium perfect gases [11], which is a Vlasov-like behavior.

We will show that the diffusion coefficient for systems with long-range interactions can be computed in the limit $N\to\infty$ with a fixed volume and renormalized interaction. At statistical equilibrium, one obtains the mean-field description typical for long-range interacting systems. Near the equilibrium, particles have an integrable motion, perturbed by the fluctuations of the mean field around its equilibrium value. This leads to the relaxation towards equilibrium. The self-consistent nature of the fluctuations (the mean field oscillates due to small particle deviations, themselves due to the mean-field fluctuations) is, however, an essential feature of this process.

In order to explore these ideas, we consider a simple toy model of long-range interacting system: the Hamiltonian mean-field model (HMF). In this framework, as a first step towards the study of the relaxation towards equilibrium, we consider the equilibrium dynamical fluctuations. We first propose an analytic computation of the autocorrelation function of the mean-field order parameter. From this result, we can derive a Fokker-Planck equation which describes the stochastic process of a particle in interaction with a bath of $N-1$ particles in equilibrium. The diffusion coefficient is then explicitly computed, from the microscopic dynamics. We finally conclude by discussing generalization to out of equilibrium situations, and more realistic models.

The Hamiltonian of the attractive HMF model [12] is

$$H=\sum_{k=1}^{N}\frac{p_k^2}{2}+\frac{1}{2N}\sum_{k,l=1}^{N}[1-\cos(\theta_k-\theta_l)]. \qquad (1)$$

Because of its simplicity, a large number of authors have considered this model and its repulsive counterpart (with the opposite sign for the potential energy). The HMF model is the "harmonic oscillator" for long-range interacting systems. We refer to Ref. [13] for a review. Let us define the magnetization $\mathbf{M}$ by $N\mathbf{M}=\Sigma_{k=1}^{N}e^{i\theta_k}$ ($\mathbf{M}=M_x+iM_y$). Because the kinetic energy per particle e_c may be exactly expressed as $2e_c=2E-1+M^2$ (E is the energy per particle), and because

M is a simple sum of N variables, the computation of the static microcanonical quantities is straightforward. For instance, we obtain the volume of the shell of the phase space, with energy E [14]: $\Omega(E) \propto_{N\to\infty} \int_0^1 dM\, B(M) \exp[NS(E,M)]$ with the entropy S given by $S(E,M)=C(M)+\ln(2E+M^2-1)/2$, where $C(M)=\ln[I_0(\psi(M))]-M\psi(M)$, I_0 is defined by $2\pi I_0(M) \equiv \int_0^{2\pi} d\theta \exp(M\cos\theta)$, and ψ as the inverse function of $d\ln I_0/dM$. The use of the saddle-point method, in the previous integral, shows that an overwhelming number of configurations have a magnetization close to the equilibrium value $M_e(E)$ defined by $\partial S(E,M_e)/\partial M=0$ [15]. This equation shows that, above the critical energy $E_c=3/4$, whereas below E_c a second-order phase transition occurs. The density in the μ-phase space (all angles and momenta are projected on a (θ,p) space) may be evaluated as $f_E(p,\theta) \propto \exp\{-\beta(E)[p^2/2-M_e(E)\cos\theta]\}$, up to a rotation of angles. These results are equivalent to the canonical ones (see Ref. [12]).

In the following we consider only energies greater than the critical one $E>E_c$. In such a case, the equilibrium is homogeneous: $M_e=0$. Then $\beta=1/(2E-1)$ and $f(p,\theta) \propto \exp(-\beta p^2/2)$. The static fluctuations of M may also be computed, from the second derivative of S with respect to M, at the equilibrium point. We obtain a Gaussian magnetization with $N\langle \mathbf{M}^\star \mathbf{M}\rangle = 2/(2-\beta)$ and $\langle \mathbf{M}^\star\mathbf{M}^\star\rangle = \langle \mathbf{M}\mathbf{M}\rangle=0$ ($\mathbf{M}^\star$ is the complex conjugate of $\mathbf{M}$). The magnetization has typical fluctuations of order $N^{-1/2}$, we thus re-scale it accordingly, by defining

$$\mathbf{m} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{i\theta_k}. \qquad (2)$$

The aim of this paper is to study the dynamical equilibrium fluctuations of this system. From the Hamiltonian (1), one obtains the equations of motion:

$$\frac{d\theta_k}{dt} = p_k \text{ and } \frac{dp_k}{dt} = \frac{N^{-1/2}}{2}[i\mathbf{m}^\star(t)e^{i\theta_k} + \text{c.c.}] \qquad (3)$$

(c.c. means the complex conjugate of the previous expression). From the motion equation (3), thanks to the smallness of the mean-field fluctuations, the motion of any particle may be treated perturbatively in the limit $N\to\infty$. We expand the variables in powers of $N^{-1/2}$: $\theta_k=\theta_{k,0}+N^{-1/2}\theta_{k,1}+\cdots$, $p_k=p_{k,0}+N^{-1/2}p_{k,1}+\cdots$, and $\mathbf{m}=\mathbf{m}_0+N^{-1/2}\mathbf{m}_1+\cdots$ (the magnetization (2) has to be self-consistent). The zero-order motion is a free ballistic one: $p_{k,0}(t)=p_k^0$ and $\theta_{k,0}(t)=\theta_k^0+p_k^0 t$, where θ_k^0 and p_k^0 are the values of θ and p for $t=0$. The expression (3) clearly shows that such a perturbative description, around this simple zero-order dynamics, will remain valid as soon as $t \ll N^{1/2}$. This expansion leads, to the first order, to $\theta_{k,1}(t) = \int_0^t du\, p_{k,1}(u)$ with

$$p_{k,1}(t) = \int_0^t du \frac{1}{2}[i\mathbf{m}_0^\star(u)e^{i(\theta_k^0+p_k^0 u)} + \text{c.c.}]. \qquad (4)$$

A peculiarity of this asymptotic expansion is that the magnetization $\mathbf{m}$ [Eq. (2)] is a sum of N variables, where $N^{-1/2}$ is the expansion parameter. A sum of N order $N^{-1/2}$ terms may be of order 1. To obtain the zero-order magnetization, we thus have to include the first-order expression of the angles $\theta_{k,1}$. We then obtain

$$\mathbf{m}_0(t) = a - 1/2 \int_0^t du \int_0^u dv [b\mathbf{m}_0^\star(v) - c\mathbf{m}_0(v)], \qquad (5)$$

where $a=N^{-1/2}\sum_{k=1}^N e^{i(\theta_k^0+p_k^0 t)}$, $b=N^{-1}\sum_{k=1}^N e^{i[2\theta_k^0+p_k^0(t+v)]}$ and $c=N^{-1}\sum_{k=1}^N e^{ip_k^0(t-v)}$. The expression (5) clearly reflects the self-consistent nature of the motion: the magnetization at time t depends on the magnetization at previous times.

From Eq. (5), we compute the magnetization autocorrelation function at leading order $\langle \mathbf{m}_0(t)\mathbf{m}_0(0)\rangle$, where the brackets denote microcanonical averages on the variables (θ_k^0, p_k^0). We first note that b and c are equal to their microcanonical average plus fluctuations of order $N^{-1/2}$. These fluctuations can be neglected at the order considered. Some lengthy computations lead to $\langle a\mathbf{m}_0\rangle = 2\exp(-t^2/2\beta)/(2-\beta)$, $\langle b\rangle=0$ and $\langle c\rangle = 2\exp[-(t-v)^2/2\beta]/(2-\beta)$ (up to order $N^{-1/2}$ corrections). From Eq. (5), we then obtain $\langle \mathbf{m}^\star(t)\mathbf{m}(0)\rangle = 2\phi(t)/(2-\beta)$ and $\langle \mathbf{m}^\star(t)\mathbf{m}^\star(0)\rangle = \langle \mathbf{m}(t)\mathbf{m}(0)\rangle=0$; where the function ϕ is given by the solution of the integral equation:

$$\phi(t) = \exp\left(-\frac{t^2}{2\beta}\right) + \frac{1}{2}\int_0^t dv\, v \exp\left(-\frac{v^2}{2\beta}\right)\phi(t-v). \qquad (6)$$

We remark that the right-hand side integral is a convolution. This makes the solution of this equation by a Laplace transform natural. We do not report the result. Whereas the first term on the right-hand side of this integral equation is due to the integrable ballistic motion of the particles, the second term reflects the self-consistent nature of the dynamics.

To have a physical insight on this autocorrelation function, we compute the asymptotic behavior of ϕ. First $\phi(t) \propto_{t\to 0} \exp[-2(2-\beta)t^2/(4\beta)]$. This approximation is obtained from Eq. (6), by a Taylor expansion. Such a Gaussian behavior for small times would be typical of a ballistic behavior. However, we note that the coefficient $(2-\beta)/(2\beta)$ is not uniquely due to the integrable zero-order dynamics, but is renormalized by the memory term. Second we obtain

$$\phi(t) \propto_{t\to\infty} A(\beta)\exp[-\gamma(\beta)t]; \gamma(\beta) = (2/\beta)^{1/2}\mathrm{F}^{-1}(\beta), \qquad (7)$$

where F^{-1} is the inverse of the function F, with $F(x)=2/[1+\sqrt{\pi}x\exp(x^2)\mathrm{erfc}(-x)]$, where erfc is the complementary error function. This exponential limit for the autocorrelation function is natural, as it corresponds to the Markovian limit for the magnetization stochastic process. The lower inset of Fig. 1 shows the relaxation constant γ as a function of β. Near the critical energy ($\beta_c=2$), the relaxation constant tends to 0. This indicates that near the critical point, the relaxation time diverges. On the contrary, for large energy, γ diverges and the relaxation time is very small. Figure 1 shows a comparison of the theoretical autocorrelation function [solving Eq. (6)] with the one obtained directly from the integration of the Hamiltonian dynamics (1).

Because the stochastic process is stationary, using the Wiener-Kinchin theorem, the spectral density of the complex magnetization may be computed from the Fourier transform

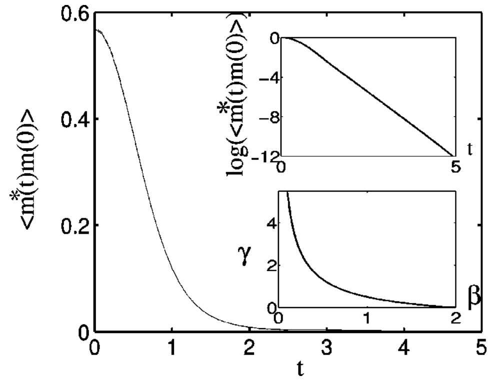

FIG. 1. The magnetization autocorrelation function: the predicted value (6) and the numerically computed value are both represented. They are indistinguishable (maximum absolute error of 3.10^{-3}). We have used $E=2.5$, $\beta=1/(2E-1)$, $N=10\,000$, and averaged over 18 samples, each one of duration $t=8000$. The upper inset shows that the exponential decay of the autocorrelation function is a good approximation times greater than 2 or 3 for these parameters. The lower inset shows the relaxation constant as a function of the inverse temperature $\gamma(\beta)$ [see Eq. (7)].

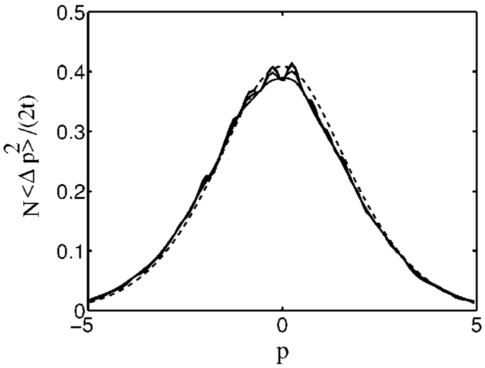

FIG. 2. The solid curve shows the mean-square displacement of a particle in function of its initial momentum, normalized by N and divided by the time $[N\langle\Delta p^2\rangle(p,t)/(2t)]$ (this is not a distribution), for four values of time: $t=10,15,20,25$; $N=10\,000$, $\beta=1/4$. As the curves are superposed for time, this shows that the motion is actually diffusive. The dashed curve represents the predicted result $D(p)$ [Eq. (8) with $D(p)=\pi S(p)/4$], no fit. This confirms the theoretical analysis, up to errors due to an incomplete statistics.

of the autocorrelation function. As the integral equation (6) is a convolution, the computation of this spectral density is easy. Defining $S(\omega)=2/\pi\int_0^\infty dt\,\cos(\omega t)\langle\mathbf{m}^\star(t)\mathbf{m}(0)\rangle$, one obtains

$$S(\omega) = \frac{16(\pi\beta/(2\pi))^{1/2}\,\exp(-\beta\omega^2/2)}{2[(2-\beta)+\beta^{3/2}\omega A(\beta^{1/2}\omega)]^2+\pi\beta^3\omega^2\exp(-\beta\omega^2)}, \tag{8}$$

where $A(x)=\exp(-x^2/2)\int_0^x du\,\exp(u^2/2)$. We have $S(\omega)\sim_{\omega\to\infty}(2\beta/\pi)^{1/2}\exp(-\beta\omega^2/2)$.

Let us now consider the diffusion of the momentum p of a single particle, where all other particles have a random angle and momentum according to the microcanonical distribution (one particle in a bath at equilibrium). Let us denote $\langle\Delta p\rangle(p,t)$ the mean displacement $p(t)-p(0)$, and $\langle\Delta p^2\rangle(p,t)$ the mean-square displacement of a particle knowing that its initial momentum $[p(0)=p]$. From Eq. (4) and the results for the autocorrelation function or for similar quantities, it is possible to compute explicitly $\langle\Delta p^2\rangle(p,t)$, at the leading order in N, for any time such that $t\ll N^{1/2}$ (perturbative description of the dynamics). The quantity $\langle\Delta p^2\rangle(p,t)$ has a transient behavior on a time scale of order 1 (the explicit computation is feasible, but not reported), followed for $1\ll t\ll N^{1/2}$, by a diffusive behavior. We then obtain $\langle\Delta p^2\rangle(p,t)\sim_{1\ll t\ll N^{-1/2}}2D(p)N^{-1}t$, with

$$D(p) = \frac{1}{2}\int_0^\infty dt\langle\mathbf{m}^\star(t)\mathbf{m}(0)\rangle\cos(pt). \tag{9}$$

This result is the equivalent of a Kubo formula. However, it states a bit more: the diffusion coefficient is there expressed as the autocorrelation of the mean field and not as the autocorrelation of the force. We note that the diffusion coefficient is proportional to the spectral density: $D(p)=\pi S(p)/4$. This is a peculiarity of this model for which the interaction is built with a cosine. An analytical expression for D is thus obtained from Eq. (8). The computation of $\langle\Delta p\rangle(p,t)$ may be done following the same procedure. Please note, however, that the $N^{-1/2}$ contribution vanishes. The lower order contribution comes from a perturbative description of the dynamics at order 2 $[p_{k,2}(t)$ and $\mathbf{m}_1(t)]$. The systematic momentum change is $N\langle\Delta p\rangle(p,t)\sim_{1\ll t\ll N^{-1/2}}[dD(p)/dp-\beta pD(p)]t$ [17]. In the following, we will also see that this result can be deduced from the stationarity of the stochastic process (or equivalently from the fact that the distribution for p tends to the equilibrium distribution). This is the equivalent of an Einstein relation. Figure 2 shows that the analytical diffusion coefficient agree with the numerically computed $N\langle\Delta p^2\rangle(p,t)/(2t)$.

We have observed a diffusive behavior for the momenta (9) with a systematic momentum drift, for $1\ll t\ll N^{1/2}$. Moreover, the mean displacement and the mean-square displacement are small as they scale like N^{-1}. These two facts are two sufficient hypotheses for the derivation of a Fokker-Planck equation (see Ref. [16]). Thus any momentum distribution function $f(p)$ evolves, at the leading order in N, through the equation

$$\frac{\partial f}{\partial t} = \frac{1}{N}\frac{\partial}{\partial p}\left[D(p)\left(\frac{\partial f}{\partial p}+\beta pf\right)\right]. \tag{10}$$

This equation is valid for time $t\gg 1$. For the derivation of the mean square and mean displacement, we have assumed $t\ll N^{1/2}$ (perturbative description). However, the previous analysis has also shown that the correlation function decays exponentially for large time. The correlation time for the force (or equivalently the magnetization) is then of order 1 and is thus much smaller than $N^{1/2}$. This is a first indication that the stochastic process may become Markovian for times much smaller than $N^{1/2}$. If it is actually so, the Fokker-Planck will be correct for any time t. We note that this equation actually converges towards the equilibrium density $P_{eq}(p) = (\beta/2\pi)^{1/2}\exp(-\beta p^2/2)$.

In this paper, we have analytically derived the autocorrelation function for the HMF model. This derivation reflects the physics of long-range interacting systems: due to this type of interaction, the autocorrelation of the mean field evolves self-consistently, as expressed by Eq. (6). We have used this result to derive analytically the diffusion of the momentum of a single particle in an equilibrium distribution, leading to a Fokker-Planck equation. A more complete study of the magnetization stochastic process, the detailed computations, and the study of this Fokker-Planck equation will be addressed in a forthcoming paper [17]. Due to the asymptotic decay of the diffusion coefficient, for large momentum, the spectrum of the linear operators of the Fokker-Planck equation has no gap between the eigenvalue corresponding to the ground state and the other eigenvalues. The present derivation is limited to states with an energy larger than the critical energy $E_c = 3/4$. Numerous studies have been devoted to homogeneous out of equilibrium, quasistationary states, with energy lower than E_c [3]. These states are very interesting as they exibit peculiar dynamical properties. For instance, the study of the correlation functions (for θ or p) [18,19] and of diffusion [19] are of particular interest. We hope to generalize, in the future, the results of this paper to such out-of-equilibrium states. The generalization of this paper's results to other long-range interacting particle models may follow the same path, and the theoretical problems linked with the divergence of some interactions at small scales (point vortices, self-gravitating systems, and plasma).

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